**Euler, Ioachimescu and the trapezium rule**

G.J.O. Jameson (*Math. Gazette* **96** (2012), 136–142)

The following results were established in a recent *Gazette* article [1, Theorems 2, 3, 4]. Given $a > 0$ and $0 < s < 1$, let

$$x_n = \sum_{r=0}^{n-1} \frac{1}{(a+r)^s} - \frac{1}{1-s}\left[(a+n)^{1-s} - a^{1-s}\right],$$

$$y_n = \sum_{r=0}^{n-1} \frac{1}{(a+r)^s} - \frac{1}{1-s}\left[(a+n-1)^{1-s} - a^{1-s}\right]$$

$$u_n = \tfrac{1}{2}(x_{n-1} + y_n) = y_n - \frac{1}{2(a+n-1)^s}$$

Then $(x_n)$ is increasing, $(y_n)$ is decreasing, and both tend to a limit, denoted by $I_s(a)$, as $n \to \infty$. Further, when $n \to \infty$,

$$n^s\,(I_s(a) - x_n) \to \tfrac{1}{2}, \qquad n^s\,(y_n - I_s(a)) \to \tfrac{1}{2}, \qquad n^{s+1}\,(I_s(a) - u_n) \to \frac{s}{12}.$$

The methods are quite lengthy, and give the appearance of being specific to this case.

Here we present a different, arguably simpler, approach to results of this sort. Without any extra effort, it delivers a more general version: $x_n$ is replaced by $\sum_{r=0}^{n-1} f(r) - \int_0^n f(x)\,dx$, where $f(x)$ is a function satisfying suitable conditions; in fact, the results gain in both clarity and simplicity when presented in this way. The first step, the convergence of $(x_n)$ and $(y_n)$, is actually no more than the familiar process leading to the existence of Euler's constant $\gamma$. The harder part is the derivation of the other limits. These are obtained in [1] using various series expansions. Our method is based instead on an estimate for the error in the trapezium rule which is well known to specialists in numerical analysis, but possibly less so to *Gazette* readers. The author hopes that it will be of interest to some of them. We describe two proofs, both quite elementary.

We consider a differentiable function $f(x)$ that is positive and decreasing for $x \geq 0$ and satisfies $f(x) \to 0$ as $x \to \infty$. Let

$$S_n = f(0) + f(1) + \cdots + f(n-1),$$

$$I_n = \int_0^n f(x)\,dx, \qquad J_r = \int_r^{r+1} f(x)\,dx,$$

$$\Delta_n = S_n - I_n, \qquad \Gamma_n = S_{n+1} - I_n,$$

so that $\Gamma_n = \Delta_n + f(n)$. Clearly, $I_n = J_0 + J_1 + \cdots + J_{n-1}$, so

$$\Delta_n = \sum_{r=0}^{n-1} [f(r) - J_r]. \tag{1}$$

The basic results about these quantities derive from the following very simple inequality. Since $f(x)$ is decreasing, we have $f(r+1) \le f(x) \le f(r)$ for $r \le x \le r+1$, hence

$$f(r+1) \le J_r \le f(r). \tag{2}$$

By (1), it follows that $\Delta_n \ge 0$. Further:

*Proposition* 1: $(\Delta_n)$ is increasing and $(\Gamma_n)$ is decreasing. Both converge to the same limit (say $L$) as $n \to \infty$, and $\Delta_n \le L \le \Gamma_n$ for all $n$.

*Proof.* We have

$$\Delta_{n+1} - \Delta_n = f(n) - J_n \ge 0,$$

$$\Gamma_{n+1} - \Gamma_n = f(n+1) - J_n \le 0.$$

So $\Gamma_n$ is bounded below (by 0) and decreasing, hence it tends to a limit, $L$, and $\Gamma_n \ge L$ for all $n$. Since $\Gamma_n - \Delta_n = f(n) \to 0$ as $n \to \infty$, $\Delta_n$ also tends to $L$.

*Example* 1: The best known example of this process is given by $f(x) = 1/(x+1)$. Then $\Delta_n = \sum_{r=1}^{n} \frac{1}{r} - \log(n+1)$, and $L$ is Euler's constant $\gamma$.

*Example* 2: Let $f(x) = 1/(a+x)^s$, where $a > 0$ and $0 < s < 1$. Then, in the notation of [1],

$$\Delta_n = x_n, \qquad \Gamma_n = y_{n+1}, \qquad L = I_s(a).$$

By (1), we have

$$L = \sum_{r=0}^{\infty} [f(r) - J_r], \tag{3}$$

and hence

$$L - \Delta_n = \sum_{r=n}^{\infty} [f(r) - J_r], \tag{4}$$

Instead of (2), we now consider the trapezium rule estimate for $J_r$, that is,

$$T_r = \tfrac{1}{2} f(r) + \tfrac{1}{2} f(r+1).$$

This is the integral of the linear function agreeing with $f(x)$ at $r$ and $r+1$. In most cases, it is a much more accurate estimate of the integral than either $f(r)$ or $f(r+1)$. If $f$ is *convex* (i.e. curving upwards), then it is obvious from a diagram that $J_r \leq T_r$; a formal proof is contained in Theorem 3 below. A sufficient condition for convexity is $f''(x) \geq 0$ for all $x$. It is satisfied by many functions of the type we are considering, including the specific examples given above. Note that

$$\sum_{r=0}^{n-1} T_r = \tfrac{1}{2}f(0) + \sum_{r=1}^{n-1} f(r) + \tfrac{1}{2}f(n) = S_n - \tfrac{1}{2}f(0) + \tfrac{1}{2}f(n).$$

To take advantage of the better approximation given by $T_r$, we introduce

$$\Lambda_n = \tfrac{1}{2}\Delta_n + \tfrac{1}{2}\Gamma_n = \Delta_n + \tfrac{1}{2}f(n).$$

Then

$$\Lambda_n = S_n + \tfrac{1}{2}f(n) - I_n = \sum_{r=0}^{n-1}(T_r - J_r) + \tfrac{1}{2}f(0),$$

hence

$$L - \Lambda_n = \sum_{r=n}^{\infty}(T_r - J_r). \tag{5}$$

Compare this with (4). For convex $f$, these formulae show at once that $(\Lambda_n)$ is increasing and $L \geq \Lambda_n$. In the notation of [1], $\Lambda_n = u_{n+1}$.

We now establish an estimate for the error in the trapezium rule, and use it to derive a pair of inequalities for $L - \Lambda_n$ which in turn will imply the limits stated in [1]. The first step is to give an estimate for the error in linear approximation to a function. This is actually the case $n = 2$ of the more general result on the polynomial interpolating a function at $n$ points (e.g. [2, p. 224]). The proof is a pleasant application of Rolle's theorem.

*Proposition* 2: Let $f$ be twice differentiable on $[a, b]$, and let $p(x)$ be the linear function agreeing with $f(x)$ at $a$ and $b$. Let $q(x) = (x - a)(b - x)$. Then, given $x$ in $(a, b)$, there exists $\xi$ in $(a, b)$ such that

$$p(x) - f(x) = \tfrac{1}{2}q(x)f''(\xi).$$

*Proof*: We prove the statement for a chosen point $x_0$ in $(a, b)$. Let $G(x) = p(x) - f(x) - kq(x)$, with $k$ chosen so that $G(x_0) = 0$, hence $p(x_0) - f(x_0) = kq(x_0)$. We have to show that $k = \tfrac{1}{2}f''(\xi)$ for some $\xi$. Now $G(a) = G(b) = G(x_0) = 0$. By Rolle's theorem, applied twice, there exists $\xi$ in $(a, b)$ such that $G''(\xi) = 0$. Now $p''(x) = 0$ and $q''(x) = -2$, so $G''(x) = -f''(x) + 2k$ for all $x$. Hence $k = \tfrac{1}{2}f''(\xi)$, as required.

*Theorem* 3: Suppose that $m \leq f''(x) \leq M$ on $[a, b]$, and let $T(f) = \tfrac{1}{2}(b-a)[f(a)+f(b)]$. Then

$$\tfrac{1}{12}m(b-a)^3 \leq T(f) - \int_a^b f(x)\,dx \leq \tfrac{1}{12}M(b-a)^3.$$

*Proof*: $T(f) = \int_a^b p(x)\,dx$, where $p(x)$ is as above. Now $q(x) \geq 0$ on $[a,b]$, so by Proposition 2,

$$\tfrac{1}{2} m q(x) \leq p(x) - f(x) \leq \tfrac{1}{2} M q(x).$$

Write $b - a = h$ and substitute $x - a = y$:

$$\int_a^b q(x)\,dx = \int_0^h y(h-y)\,dy = \left[\tfrac{1}{2}hy^2 - \tfrac{1}{3}y^3\right]_0^h = \tfrac{1}{6}h^3.$$

So $\int_a^b [p(x) - f(x)]\,dx$ lies between $\frac{1}{12}mh^3$ and $\frac{1}{12}Mh^3$.

*Note*: We pause to sketch a second, equally attractive, proof of Theorem 3. Write $c = \frac{1}{2}(a+b)$ and $\int_a^b f(x)\,dx = I(f)$. Integration by parts gives

$$\begin{aligned}
\int_a^b (x-c)f'(x)\,dx &= \Big[(x-c)f(x)\Big]_a^b - \int_a^b f(x)\,dx \\
&= \tfrac{1}{2}(b-a)[f(b)+f(a)] - I(f) \\
&= T(f) - I(f).
\end{aligned}$$

Now integrate by parts the other way round! With $q(x)$ as above, we have $q'(x) = a+b-2x = 2(c-x)$, so we can use $-\frac{1}{2}q(x)$ as the antiderivative of $x - c$. We obtain

$$\begin{aligned}
T(f) - I(f) &= \Big[-\tfrac{1}{2}q(x)f'(x)\Big]_a^b + \tfrac{1}{2}\int_a^b q(x)f''(x)\,dx \\
&= \tfrac{1}{2}\int_a^b q(x)f''(x)\,dx.
\end{aligned}$$

since $q(a) = q(b) = 0$. Again the integrand lies between $\frac{1}{2}mq(x)$ and $\frac{1}{2}Mq(x)$, and the statement follows as before.

Clearly, if $f''(x)$ is decreasing, then Theorem 3 applies with $M = f''(a)$ and $m = f''(b)$.

Hence for $T_r$ and $J_r$ defined as above, if $f''(x)$ is decreasing, then

$$\tfrac{1}{12}f''(r+1) \leq T_r - J_r \leq \tfrac{1}{12}f''(r).$$

We can now state our basic result on $L - \Lambda_n$.

*Theorem* 4: Suppose that

(a) $f(x)$ is positive and decreasing for $x \geq 0$,

(b) $f(x)$ and $f'(x)$ tend to 0 as $x \to \infty$,

(c) $f''(x)$ is positive and decreasing for $x \geq 0$.

Then

$$-\tfrac{1}{12}f'(n+1) \leq L - \Lambda_n \leq -\tfrac{1}{12}f'(n-1).$$

(Note that $f'(x)$ is negative.)

*Proof*: By (5) and Theorem 3, provided that $\sum_{r=1}^{\infty} f''(r)$ converges, we have

$$\tfrac{1}{12} \sum_{r=n+1}^{\infty} f''(r) \leq L - \Lambda_n \leq \tfrac{1}{12} \sum_{r=n}^{\infty} f''(r).$$

Since $f''(x)$ is decreasing, (2) (applied to successive intervals) gives

$$\sum_{r=n}^{\infty} f''(r) \leq \int_{n-1}^{\infty} f''(x)\, dx = -f'(n-1)$$

and

$$\sum_{r=n+1}^{\infty} f''(r) \geq \int_{n+1}^{\infty} f''(x)\, dx = -f'(n+1).$$

The list of conditions on $f(x)$ might seem rather long, but it is very easily seen that they are all satisfied by $1/(a+x)^s$ (where $s > 0$). Actually, with a bit of effort, one can show that the condition $f'(x) \to 0$ follows from the others.

*Example* 3: To apply this to $\gamma$, take $f(x) = 1/(x+1)$, and replace $n$ by $n-1$ in Theorem 4. Then $\Lambda_{n-1} = \sum_{r=1}^{n-1} \frac{1}{r} + \frac{1}{2n} - \log n$, and we deduce that $\gamma = \Lambda_{n-1} + R_{n-1}$, where

$$\frac{1}{12(n+1)^2} \leq R_{n-1} \leq \frac{1}{12(n-1)^2}.$$

Even with $n$ quite small, this gives a good approximation to $\gamma$. For example, when $n = 4$, the resulting lower and upper bounds for $\gamma$ are 0.57537 and 0.58130.

The same work applies, rather more directly, to the estimation of the tail of a convergent series. Suppose that $f$ satisfies (a), (b), (c) and that $\int_1^{\infty} f(x)\, dx$ is convergent. Then, by (2), $\sum_{n=1}^{\infty} f(n)$ is convergent (this is the "integral test for convergence"). Write

$$S_n^* = \sum_{r=n}^{\infty} f(n), \qquad I_n^* = \int_n^{\infty} f(x)\, dx.$$

It is a familiar fact that $I_n^*$ approximates $S_n^*$ in some sense. We can now describe this approximation rather accurately. Clearly, $\sum_{r=n}^{\infty} T_r = S_n^* - \frac{1}{2} f(n)$ and $\sum_{r=n}^{\infty} J_r = I_n^*$. So the proof of Theorem 4 gives:

*Theorem* 5: Under these conditions, $S_n^* = I_n^* + \frac{1}{2} f(n) + R_n$, where $-\frac{1}{12} f'(n+1) \leq R_n \leq -\frac{1}{12} f'(n-1)$.

*Example* 4: Let $f(x) = 1/x^s$, where $s > 1$. The sum of the series $\sum_{n=1}^{\infty} 1/n^s$ is the Riemann zeta function $\zeta(s)$. In the notation of Theorem 5,

$$S_n^* = \frac{1}{(s-1)n^{s-1}} + \frac{1}{2n^s} + R_n,$$

where

$$\frac{s}{12(n+1)^{s+1}} \le R_n \le \frac{1}{12(n-1)^{s+1}}.$$

*Note*: If $f''(x)$ is convex, then a refinement of the second proof of Theorem 3 (which we will not describe here) leads to $T_r - J_r \le \frac{1}{12}[f'(r+1) - f'(r)]$. This means that the $f'(n-1)$ in Theorem 4 can be replaced by $f'(n)$, so that the $n-1$ can be replaced by $n$ in Examples 3 and 4.

Finally, we return to the limits stated at the beginning. We assume two further conditions which are clearly satisfied by $1/(a+x)^s$. Again, we will not take any trouble investigating the extent to which some of the conditions are implied by the others.

*Theorem* 6: Let $f(x)$ be as in Theorem 4, and assume further:

$$\text{(d)}\ \frac{f'(x+1)}{f'(x)} \to 1 \quad \text{as } x \to \infty, \qquad \text{(e)}\ \frac{f'(x)}{f(x)} \to 0 \quad \text{as } x \to \infty.$$

Then, when $n \to \infty$,

$$\frac{L - \Lambda_n}{f'(n)} \to -\tfrac{1}{12},$$

$$\frac{L - \Lambda_n}{f(n)} \to 0, \qquad \frac{L - \Delta_n}{f(n)} \to \tfrac{1}{2}, \qquad \frac{\Gamma_n - L}{f(n)} \to \tfrac{1}{2}.$$

*Proof.* The first statement follows at once from Theorem 4 and (d), since (d) also implies that $f'(x-1)/f'(x) \to 1$ as $x \to \infty$. Condition (e) now gives the second statement. The last two statements follow at once, since $\Delta_n = \Lambda_n - \frac{1}{2}f(n)$ and $\Gamma_n = \Lambda_n + \frac{1}{2}f(n)$.

This illustrates nicely the fact that $\Lambda_n$ is a better approximation to $L$ than $\Delta_n$ or $\Gamma_n$.

These limits reproduce the ones from [1]. Consider, for example, the first one. With $f(x) = 1/(a+x)^s$ and the notation of [1], we have

$$\frac{L - \Lambda_{n-1}}{f'(n-1)} = -(I_s(a) - u_n)\,\frac{(a+n-1)^{s+1}}{s},$$

so that

$$n^{s+1}(I_s(a) - u_n) = -\frac{sn^{s+1}}{(a+n-1)^{s+1}}\,\frac{L - \Lambda_{n-1}}{f'(n-1)} \to \frac{s}{12} \quad \text{as } n \to \infty.$$

A further limit stated in [1] is $\lim_{n\to\infty} n^{s+1}(z_n - I_s(a)) = s/6$, where $z_n = \frac{1}{2}(x_n + y_n)$ (Theorem 3). Recall that $y_n = \Gamma_{n-1}$. We leave it as an exercise for the reader to show that under our conditions, $\lim_{n\to\infty}(\Gamma_{n-1} - \Gamma_n)/f'(n) = -\frac{1}{2}$ and to derive the limit just stated.

*References*


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